SELF-SIMILAR PROBLEMS OF LONGITUDINAL WAVE PROPAGATION IN NONLINEAR MEDIA

V. P. Yastrebov

UDC 534.1

The regularities of one-dimensional longitudinal wave propagation in nonlinear media are described in [1-5]. The substantial influence of the kind of dependence between the stress and strain in the material on the nature of the wave process and the possibility of the occurrence of discontinuous solutions (shocks) are noted [1, 2]. Self-similar motions for a quite general form of the stress—strain dependence were investigated in [3] for an instantaneously applied constant stress given on the boundary of the medium. Certain self-similar problems were studied in [6]; however, this work contains inaccuracies.

Conditions are indicated in this paper for which wave propagation problems will be self-similar. Such conditions assure assignment of a power-law dependence between the stress and strain in the material, and a power-law change with time in the quantities governing the boundary conditions. Appropriate solutions are found. The method of characteristics is used extensively in seeking the solutions. A combination of this method with the self-similar representation of the solution permits writing it in convenient form. In the case of discontinuous solutions, application of the method of characteristics if fraught with difficulties. Consequently, traditional methods that are based on the properties of self-similar problems must be used. This would permit obtaining new results and giving an estimate of certain assumptions imposed in application of the method of characteristics.

1. Let us examine the question of the conditions under which the solution is self-similar. The one-dimensional medium is treated as a homogeneous rectilinear rod of constant unit cross section. Let u be the displacement of rod particles along its axis, p the stress in the transverse section, x the coordinate measured to the right along the length of the rod from its left endface superposed at the origin, t is the time, $\varepsilon = \partial u/\partial x$ is the strain, and ρ is the material density. Tensile stresses are considered positive. The stress and strain are related by the dependence

$$p = Ef(\varepsilon), \tag{1.1}$$

where E is a constant with the dimensionality of the stress, and $f(\varepsilon)$ is a dimensionless function of ε. The rod state of strain is described by the equation [1]

$$\partial^2 u/\partial t^2 = a^2(\varepsilon)\partial^2 u/\partial x^2,\ a^2(\varepsilon) = E\rho^{-1}df/d\varepsilon. \tag{1.2}$$

The solution of the problem in the case of self-similar motion is sought in the form [7-9]

$$u = u_* t^\alpha \varphi(\xi); \tag{1.3}$$

$$p = p_* t^\delta \psi(\xi), \tag{1.4}$$

where $\varphi(\xi)$ and $\psi(\xi)$ are dimensionless functions of the dimensionless variable

$$\xi = x/(bt^\beta); \tag{1.5}$$

α, β, and δ are exponents which are unknown as yet, and u_*, p_*, and b are dimensional constants. Substituting (1.3) into (1.2) with (1.5) taken into account yields

$$E\rho^{-1}b^{-2}t^{-2\beta+2}(df/d\varepsilon)\varphi'' = \beta^2\xi^2\varphi'' - \beta(2\alpha - \beta - 1)\xi\varphi' + \alpha(\alpha - 1)\varphi. \tag{1.6}$$

Here the primes denote differentiation with respect to the variable ε. The derivative $df/d\varepsilon$ is a function of the strain ε, which is, in turn, determined by the formula

$$\varepsilon = \partial u/\partial x = u_* b^{-1} t^{\alpha-\beta}\varphi'. \tag{1.7}$$

Leningrad. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 4, pp. 118-128, July-August, 1985. Original article submitted April 23, 1984.

0021-8944/85/2604-0559$09.50

Necessary for the possibility of a self-similar solution to exist for an arbitrary form of the function $f(\varepsilon)$ is that the time not enter explicitly in (1.6) and (1.7). This condition is satisfied when the exponents of the t in (1.6) and (1.7) are zero. Hence, $\alpha = \beta = 1$ is obtained.

If (1.7) is explicitly independent of the time, then according to (1.1) the stress should also be explicitly independent of t. In this case, δ equals zero in (1.4), and we have $p = p_*\psi(0) = \text{const}$ on the rod endface ($\xi = 0$). Therefore, for an arbitrary dependence between p and ε the self-similar solution can be obtained for a constant stress (instantaneously applied) or strain on the rod endface [3].

A further extension of the class of self-similar motions is possible by giving the dependence (1.1) a form which would permit deducing the time from the derivative $df/d\varepsilon$ in (1.6) when substituted into (1.7). For this it is sufficient to take the power-law dependence

$$p = Ef(\varepsilon),\ f(\varepsilon) = |\varepsilon|^{\mu} \operatorname{sign} \varepsilon,\ \mu > 0, \tag{1.8}$$

where μ is a given exponent. Substituting the new value of $df/d\varepsilon$ into (1.6) with (1.7) taken into account, and equating the exponent for t to zero, we obtain an equation for α and β:

$$(1 - \mu)\alpha + (1 + \mu)\beta = 2. \tag{1.9}$$

Substituting (1.4) and (1.7) into (1.8) yields

$$\psi = (E/p_*)(u_*/b)^{\mu}\, t^{(\alpha-\beta)\mu-\delta}\, |\varphi'|^{\mu} \operatorname{sign}\varphi', \tag{1.10}$$

from which there follows

$$(\alpha - \beta)\mu - \delta = 0. \tag{1.11}$$

The third equation relating the exponents α, β, and δ is obtained from considering the boundary conditions. We assume that a compressive stress is applied to the rod at the left endface for $x = 0$, while it extends without limit to the right (for a rod of finite length the self-similarity is spoiled as a rule [10]). The structure of the expression (1.4) indicates that the stress in the endface section ($\xi = 0$) should change according to the power law

$$p = -p_0 t^{\lambda},\ \lambda \geqslant 0, \tag{1.12}$$

where p_0 and λ are given constants. Equating (1.4) and (1.12) for $\xi = 0$, we obtain

$$\delta = \lambda,\quad \psi(0) = -p_0/p_*. \tag{1.13}$$

Solving (1.9), (1.11), and the first equality in (1.13) jointly, we find

$$\alpha = 1 + [\lambda(\mu + 1)/(2\mu)],\ \beta = 1 + [\lambda(\mu - 1)/(2\mu)],\ \delta = \lambda. \tag{1.14}$$

The constants u_*, p_*, and b should be expressed in terms of the governing parameters of the problem E, ρ, p_0. Using dimensional analysis of the quantities [7], we can form the following combinations:

$$b = (E/\rho)^{1/2}(E/p_0)^{(1-\mu)/(2\mu)}; \tag{1.15}$$

$$u_* = (E/\rho)^{1/2}(p_0/E)^{(\mu+1)/(2\mu)}; \tag{1.16}$$

$$p_* = p_0. \tag{1.17}$$

Substituting these values into (1.6) and (1.10), we have the equations

$$[\beta^2\xi^2 - \mu|\varphi'|^{\mu-1}]\varphi'' - \beta(2\alpha - \beta - 1)\xi\varphi' + \alpha(\alpha - 1)\varphi = 0; \tag{1.18}$$

$$\psi = |\varphi'|^{\mu} \operatorname{sign}\varphi'. \tag{1.19}$$

The second equality in (1.13) and Eqs. (1.17), (1.19) yields the boundary condition at $\xi = 0$:

$$\psi(0) = \varphi'(0) = -1, \tag{1.20}$$

The differential equations obtained permit solution of the problem formulated. However, in many cases the methods based on using the characteristics of equations of hyperbolic type afford the possibility of reaching the result more simply; hence, they are utilized extensively below. The exception is the important case of discontinuous solutions for which it is necessary to turn to direct integration of (1.18) and (1.19).

For discontinuous solutions it is also necessary to take account of conditions resulting from general theorems of mechanics. Let us construct these conditions. According to the theorem on the change in momentum of a mechanical system, the momentum acquired by a rod should equal the impulse of the external pressure (1.12) acting on the left endface of the rod in the time interval under consideration. On the basis of a theorem on the change in energy, the sum of the kinetic and potential energies of the rod equals the work of the pressure (1.12) applied to the rod enface [11]. The mathematical description of the conditions resulting from these two theorems takes the form

$$\int_0^{\xi} (\alpha\varphi - \beta\xi\varphi')\, d\xi = 1/(\lambda + 1); \tag{1.21}$$

$$\int_0^{\xi} [0.5\,(\alpha\varphi - \beta\xi\varphi')^2 + (\mu + 1)^{-1}\,|\varphi'|^{\mu+1}]\, d\xi = \alpha\varphi\,(0)/(\alpha + \lambda), \tag{1.22}$$

after substitution of (1.3) and (1.12) and utilization of (1.14)-(1.17), where $\varphi(0)$ is the value of the function $\varphi(\xi)$ for $\xi = 0$, and the upper limit of integration corresponds to the extent of the rod involved with strains.

2. Let us consider the wave motion in a rod subjected to the stress (1.12) by assuming that it is at rest at t = 0. In this section we limit ourselves to the consideration of the case $\lambda \neq 0$ corresponding to a monotonic growth in the stress at the rod endface. First, new dimensionless variables

$$x_* = x/L, \quad \tau = t/T, \quad U = u/L, \quad V = vT/L$$

are introduced, where L and T are quantities with the dimensionality of length and time

$$L = (E/\rho)^{1/2}(E/p_0)^{1/\lambda}, \quad T = (E/p_0)^{1/\lambda}, \tag{2.1}$$

v is the velocity of the rod particles $v = \partial u/\partial t$. In the new variables (1.2) becomes

$$\partial^2 U/\partial\tau^2 = a_*^2(\varepsilon)\,\partial^2 U/\partial x_*^2, \quad a_*^2(\varepsilon) = df/d\varepsilon. \tag{2.2}$$

The relationships

$$\xi = x_*/\tau^{\beta}, \quad U = \tau^{\beta}\varphi(\xi), \quad p/E = \tau^{\delta}\psi(\xi), \quad \varepsilon = \partial U/\partial x_*; \tag{2.3}$$

$$V = \partial U/\partial\tau = \tau^{\alpha-1}(\alpha\varphi - \beta\xi\varphi') \tag{2.4}$$

can also be obtained. We write the boundary condition (1.12) in the form $p_0/E = -\tau^{\lambda}{}_0$, or according to (1.8), in terms of the strain

$$\varepsilon_0 = -\tau_0^{\lambda/\mu}, \tag{2.5}$$

where the subscript 0 denotes the value of ε for x = 0 and of the variable τ ("dimensionless time") when it is in the boundary condition for x = 0.

Let us consider the characteristics of the differential equation (2.2) [1, 4, 5, 12]. Two families of characteristics defined by the equations

$$d\tau/dx_* = \pm 1/a_*(\varepsilon) \tag{2.6}$$

can be constructed in the plane with coordinate axes x_* and τ. The conditions

$$V = \int_0^{\varepsilon} a_*(\varepsilon)\, d\varepsilon + C_1, \quad V = -\int_0^{\varepsilon} a_*(\varepsilon)\, d\varepsilon + C_2 \tag{2.7}$$

are satisfied on the characteristics. The plus and minus signs in these formulas are conferred on characteristics of positive and negative slope. The constants C_1 and C_2 correspond to the two mentioned mentioned families of characteristics and have values on each characteristic.

Later, in addition to the variable τ, the variable

$$\theta = \tau/\tau_0 \tag{2.8}$$

will be used. As was noted, τ_0 corresponds to the value of τ at $x = 0$. This quantity can be considered as a parameter denoting any characteristic taking the origin on the axis 0τ for an appropriate value of τ_0. Shown in Fig. 1 (for $\mu < 1$) are the solid and dashed lines of the characteristics of positive and negative slope marked by values of τ_0. Graphs of the dependence between x_* and τ for certain values of $\xi = \text{const}$ are superposed by dash-dot lines. In considering a specific characteristic ($\tau_0 = \text{const}$) it is seen that besides the values of x_* and τ definite values of the quantities ξ and θ correspond to each of its points, and can be selected as new variable to describe the equations of the characteristics.

The form of the solution of the stress wave propagation problem depends substantially on (1.8). Its graphical display in the coordinate axes ε, $f(\varepsilon)$ can yield convexity or concavity with respect to the axis ε, which is governed by the sign of the derivative $d^2f/d\varepsilon^2$ or the value of μ in (1.8) (μ less or greater than one) [1, 2, 4, 5].

Let us consider the case $\mu < 1$. The characteristics of positive slope form a bundle of diverging lines in the coordinate plane $x_*0\tau$ in this case (see Fig. 1). The characteristics of negative slope will tend asymptotically to the $0x_*$ axis for the dependence (1.8) taken. On the abscissa axis ($\tau = 0$), by virtue of the zero initial conditions $\varepsilon = 0$ and $V = 0$ hold and, consequently, the constant C_2 in the second equation in (2.7) is zero for all the negative-slope characteristics, i.e.,

$$V = -\int_0^{\varepsilon} a_*(\varepsilon)\, d\varepsilon. \tag{2.9}$$

This equation should be satisfied in all planes of the variables x_*, τ, including on all positive-slope characteristics. For (2.9) to be compatible with the first equation in (2.7) that yields the condition on these characteristics, it must be considered that

$$C_1 = -2\int_0^{\varepsilon} a_*(\varepsilon)\, d\varepsilon = \text{const},$$

which can be satisfied only if ε and V are, respectively, constant on each positive characteristic [1, 4, 5].

We integrate Eq. (2.6) of the positive-slope characteristics by taking into account that ε is constant and equal to ε_0 in (2.5), i.e., the value for $x_* = 0$:

$$\tau - \tau_0 = x_*/a_*(\varepsilon_0). \tag{2.10}$$

Further manipulations, in which the second expression in (2.2), and formulas (1.8), (2.5), (1.14), (2.8) are used, successively yield from (2.10)

$$\theta - 1 = \xi\theta^{\beta}/\sqrt{\mu}. \tag{2.11}$$

Since the deformation is conserved invariant along the positive-slope characteristic, the stress that equals $p/E = -\tau^{\lambda}{}_0$ by virtue of boundary condition (1.12) also remains constant. Equating this value to the third equation in (2.3), and taking account of (1.14), we obtain a formula determining the desired solution of (1.4):

$$\psi(\xi) = -1/\theta^{\lambda}. \tag{2.12}$$

It is necessary to move along the characteristic in computing (2.12). This is equivalent to the fact that the quantity θ will be expressed in terms of ξ by using the dependence (2.11).

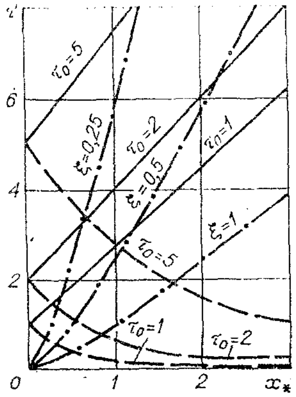

Fig. 1

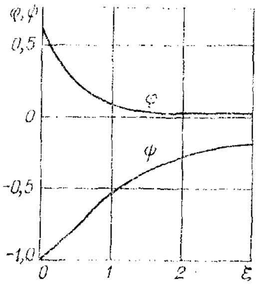

Fig. 2

Integrating in (2.9), we obtain

$$V = -2\sqrt{\mu}(\mu+1)^{-1}|\varepsilon|^{(\mu+1)/2}\operatorname{sign}\varepsilon.$$

The quantities that enter here on each positive-direction characteristic remain constant and equal to their values for $x_* = 0$ (2.5), which yields with (1.14) taken into account

$$V = 2\sqrt{\mu}(\mu+1)^{-1}\tau_0^{\alpha-1}. \tag{2.13}$$

To obtain the function $\varphi(\xi)$, the value (2.12) is substituted into (1.19) and integration is performed:

$$\varphi(\xi) = -\int\theta^{-\lambda/\mu}d\xi = -\sqrt{\mu}\,\theta^{-\alpha}[\theta(1-\beta)(1-\alpha)^{-1} - \beta/\alpha] + C, \tag{2.14}$$

where C is an arbitrary constant of integration. The variable ξ was replaced in the integration by the variable θ by using (2.11). To find C, (2.14) is substituted into (2.4), which should equal the constant value (2.13) on each positive-slope characteristic. It can be proved that C = 0 should be taken to satisfy such an equality. By using (1.14) we reduce (2.14) to the form

$$\varphi(\xi) = 2\sqrt{\mu}(\mu+1)^{-1}\alpha^{-1}\theta^{-\alpha}[1 + 0.5\alpha(1-\mu)(\theta-1)]. \tag{2.15}$$

Here, as in (2.12), the variable θ should be considered as a function of ξ defined by (2.11). In executing practical computations it is convenient to find the dependence of the functions (ξ) and $\psi(\xi)$ on θ, and then to determine the appropriate values of ξ from (2.11).

Graphs of the quantities $\varphi(\xi)$ and $\psi(\xi)$, computed for $\mu = 1/3$ and $\lambda = 1/5$, are represented in Fig. 2. The characteristics corresponding to this case are constructed in Fig. 1. Direct numerical integration of the differential equations (1.18) and (1.19) on a computer yielded results in complete agreement with an analytic computation. Conditions (1.21) and (1.22) are satisfied.

The solution obtained yields instantaneous propagation of the perturbations along the rod. This is explained by the fact that small deformations correspond to small times as the stress grows smoothly on the rod endface. For small deformations the stress–strain dependence (1.8) has a large derivative $df/d\varepsilon$ for $\mu < 1$, on which the quantity $a_*(\varepsilon)$ in (2.2) that characterizes the rate of perturbation propagation depends directly. In the limit as $\varepsilon \to 0$, there holds $a_*(\varepsilon) \to \infty$, i.e., the perturbation velocities are infinite; however, their amplitudes turn out to be infinitesimal here.

Let us turn to the case $\mu > 1$ ($d^2f/d\varepsilon^2 > 0$). As before, we consider $\lambda \neq 0$. If we move along the axis $O\tau$ of the coordinate plane $x_*O\tau$, then, according to (2.6), the slope of the positive characteristics will diminish continuously at the axis for $\mu > 1$. The characteristics here form a converging family of lines. This means that the perturbations occurring at the left end of the rod ($x = 0$) in a later period will overtake the perturbations being propagated earlier by producing conditions for the occurrence of a discontinuous solution (shock wave). Positive and negative direction characteristics are represented by, respectively, solid and dashed lines in Fig. 3 for $\mu > 1$. Also shown there by dash-dot lines are dependences between x_* and τ for certain values of ξ.

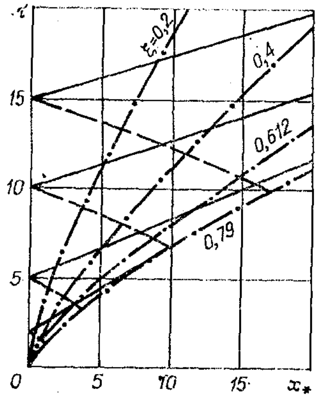

Fig. 3

We mentally draw a third coordinate axis U perpendicularly to the plate $x_*O\tau$ and which will denote the elevation of points of the integral surface $U(x_*, \tau)$ of the initial differential equation (2.2) above the plane $x_*O\tau$. Characteristic lines whose projections on the plane $x_*O\tau$ yield the characteristics examined above [12] can be drawn on the integral surface. The positive direction characteristics that are infinitely close together intersect at points forming a line that is the envelope of the family of characteristics. This envelope is the projection of the cuspidal edge of the integral surface on the plane $x_*O\tau$. Above the part of the $x_*O\tau$ plane located between the axis Ox_* and the envelope, there is no integral surface and no perturbations. The derivatives of $U(x_*, \tau)$ cannot be continuous in the neighborhood of the cuspidal edge [12]. The envelope of the characteristics or any curve on which a discontinuity of the solution occurs can be considered as the boundary curve for the negative-slope characteristics. The values of the strains and the rates are not known on it in advance; consequently, the magnitude of the arbitrary constant C_2 in the condition (2.7) on the negative-slope characteristic remains undetermined. In this case, the method of proof used above to establish the constancy of the deformation on the positive-slope characteristics does not pass. Therefore, it is impossible to assert that these characteristics will certainly be straight lines and (2.9) turns out to be correct on them. The solution in which the characteristics will be rectilinear should be considered as one of those possible.

Because the right sides of the equations of the characteristics (2.6) remain undetermined, utilization of the method of characteristics is fraught with difficulty. Consequently, the solution is sought by numerical integration of the differential equations (1.18) and (1.19). The characteristics can be found simultaneously during the computation. To do this their equations (2.6) are manipulated as follows. The value $\varepsilon = \partial U/\partial x_* = \tau^{\alpha-\beta}\varphi'$ is substituted, and the differential dx_* is expressed in terms of the differentials $d\xi$ and $d\tau$ by using the first equation in (2.3). Equations with separable variables are obtained, which after integration and subjection to the initial conditions (for $\xi = 0$, $\tau = \tau_0$ or $\theta = 1$), yield the following equations for the characteristics in general form:

$$\theta = \exp\left[\pm \int_0^{\xi} d\xi/(\sqrt{\mu}\,|\varphi'|^{(\mu-1)/2} \mp \xi\beta)\right].$$

The upper signs in this expression refer to the positive-slope, and the lower to the negative-slope characteristics.

Equation (1.18) has a singular point ξ_1 at which φ'' tends to infinity, while φ'' has a discontinuity. This point is determined by the disappearance of the coefficient of φ'', i.e., by satisfaction of the equality

$$\beta\xi_1 = \sqrt{\mu}|\varphi'(\xi_1)|^{(\mu-1)/2}. \tag{2.16}$$

The value ξ_1 corresponds to the location of the envelope of the characteristics whose equation on the $x_*O\tau$ plane has the form $x_* = \xi_1\tau^{\beta}$. The envelope is shown in Fig. 3 by dash-dot line $\xi = 0.79$.

Integration of (1.18) was by a numerical method of the MIR-2 and Élektronika-60M computers. The factorization method was used from the zeroth value of ξ (from the left end) for which just the one boundary condition (1.20) is known. The other condition of $\varphi(0)$ for $\xi = 0$ was given arbitrarily and refined by the results of the factorization to the right end. By giving a set of values of $\varphi(0)$ at the left and, a set of solutions has a discontinuity of the derivative at certain points can be obtained, among which one physically allowable should be selected [12]. The following should be taken into account for the selection of this solution: the natural condition about the displacement on the shock front being zero, which, taking (1.3) into account, is written in the form

$$\varphi(\xi_*) = 0 \tag{2.17}$$

(ξ_* is the value of the variable ξ on the wave front); theorems about the change in momentum (1.21) and the change in energy (1.22); mass, momentum, and energy conservation laws directly on the front of the discontinuity. If it is considered that the rod is at rest in front of the wave front, then the mass conservation law and the theorem on the change in momentum yield a condition on the front of the discontinuity [4] $p = -\rho D \partial u/\partial t$ (D is the propagation velocity of the front of the discontinuity). The value of D can be determined from (1.5), where the quantity ξ corresponds to the location of the front $\xi = \xi_*$:

$$D = dx/dt = \xi_* b \beta t^{\beta-1}.$$

Using this value, as well as (1.3), (1.4), (1.14)-(1.17), (1.19), and (2.17), the condition on the front of the discontinuity can be reduced to the form

$$\beta\xi_* = |\varphi'(\xi_*)|^{(\mu-1)/2}. \tag{2.18}$$

The equations of the characteristics and the integrals in (1.21) and (1.22) were computed and condition (2.18) was also verified during execution of the calculations on a computer simultaneously with integration of (1.18) and (1.19). The computations were performed for $\mu = 3$ and $\lambda = 1$. In the first variant of the computation it is assumed that the singularity ξ_1 of (1.18), determined by (2.16), is simultaneously the point of location of the shock front. Consequently, the condition (2.17) that the displacement equals zero at this point is taken as the basis for performing the numerical factorizations. The results obtained show that the theorem about the change in energy is satisfied with a certain error (8.5%), while the theorem about the change in momentum and condition (2.18) are not satisfied. Moreover, the momentum acquired by the rod turns out to be greater than the pressure pulse applied to the endface, which contradicts the physical meaning. Let us note that (2.18), formulated for the front of the discontinuity, cannot generally be satisfied at the singularity ξ_1, whose location is determined by (2.16), since these formulas differ by a factor of $\sqrt{\mu}$, making them incompatible at one point. The positive-direction characteristics for the case considered are somewhat curved in the plane of the variables x_* and τ. Therefore, it is impossible to consider this version of the solution physically allowable.

In the second version, the location of the front ξ_* is not related to the singular point ξ_1 but is determined from the condition that the displacement on the front vanishes simultaneously with compliance with the theorem on the change in momentum (1.21). Computations showed that the wave front is closer to the left end of the rod than the singular point. The condition (2.18) on the front is satisfied here. An energy loss occurs, as is noted in [2, 3]. The results of computing the quantities $\varphi(\xi)$ and $\psi(\xi)$ are represented in Fig. 4 by solid curves. The part of the solution not realizable, which lies between the wave front and the singular point ξ_1, is shown by the dash-dot lines. Figure 3 yields a portrait of the characteristics of the second version of the computation ($\mu = 3$, $\lambda = 1$). The dash-dot line for $\xi = 0.612$ in it determined the wave front location. The envelope of the characteristics corresponds to the value $\xi = 0.79$. The positive slope characteristics are quite close to straight lines. The solution obtained should be considered allowable.

Let us examine the question of the possibility of constructing a solution for $\mu > 1$ under the assumption that the positive-direction characteristics are straight lines. The solution has the same form as for $\mu < 1$, (2.11), (2.12), (2.15). The singular point of (1.18) has the value $\xi_1 = \sqrt{\mu}(\beta - 1)^{\beta-1}\beta^{-\beta}$ in this case. Computations performed for $\mu = 3$, $\lambda = 1$ show that the conditions (1.21), (2.17), (2.18) are satisfied for $\xi = 0.687$, 0.555, 0.621. This means that they cannot be satisfied simultaneously for certain values of ξ_* governing the location of the front of the discontinuity. Consequently, the solution based on the assumption of rectilinearity of the characteristics can be considered only approximate. The functions $\varphi(\xi)$ and $\psi(\xi)$ of this solution are shown in Fig. 4 by dashes. The location of the shock

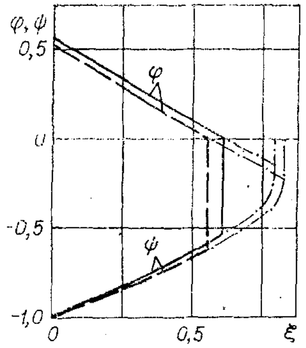

Fig. 4

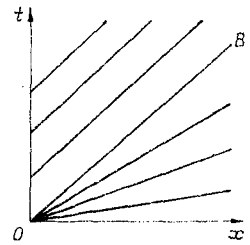

Fig. 5

is referred provisionally to the point where the displacement (2.17) vanishes. The unrealizable part of this solution lies ahead of the wave front up to the singular point ξ_1. It is shown by dash-dot lines. The results of the approximate and exact solutions turn out to be close. Sufficiently good agreement is obtained also for another computational case: $\mu = 3$, $\lambda = 1.5$.

3. We obtain the solution for an instantaneously applied constant stress (1.12) $\lambda = 0$ investigated in [1, 3, 5] for an arbitrary dependence (1.1). For $\lambda = 0$ the formulas (2.1) are meaningless; hence, the original variables x and t are used in examining the characteristics. The equations of the characteristics and the conditions on them, analogous to (2.6) and (2.7), are written in the form

$$dt/dx = \pm 1/a(\varepsilon), \quad v = \pm \int_0^{\varepsilon} a(\varepsilon)\, d\varepsilon + C_{1,2}. \tag{3.1}$$

The quantities (1.14) $\alpha = 1$, $\beta = 1$, $\delta = 0$ for which (1.18) dissociates into two equations

$$\varphi'' = 0, \quad \xi^2 - \mu|\varphi'|^{\mu-1} = 0$$

correspond to the value $\lambda = 0$. Their integrals have the form

$$\varphi = A_1 + A_2\xi, \quad \varphi = \pm(\mu - 1)(\mu + 1)^{-1}\mu^{1/(1-\mu)}\xi^{(\mu+1)/(\mu-1)} + A_3, \tag{3.2}$$

where A_1, A_2, and A_3 are arbitrary constants of integration.

Let us consider the case $\mu < 1$. We use the method of characteristics. Using the method analogous to that applied in the previous section for $\mu < 1$, it can be shown that the positive-slope characteristics are straight lines, while the strains and the velocities on them are constants [1, 4, 5], and the second equations in (3.1) are transformed into

$$v = -\int_0^{\varepsilon} a(\varepsilon)\, d\varepsilon = -\sqrt{E/\rho}\, 2\sqrt{\mu}(\mu + 1)^{-1}|\varepsilon|^{(\mu+1)/2} \operatorname{sign} \varepsilon \tag{3.3}$$

in the whole domain of the variables x and t. For $x = 0$, by virtue of the boundary condition for $\lambda = 0$ and Eq. (1.8), the strain equals the constant value

$$\varepsilon = \varepsilon_0 = -(p_0/E)^{1/\mu}. \tag{3.4}$$

We integrate the equation for the positive-slope characteristics (3.1) by taking into account that the deformation is constant on them and equal to (3.4):

$$t - t_0 = x/a(\varepsilon_0). \tag{3.5}$$

Here t_0 is the value of the time referred to the Ot axis of the xOt plane (Fig. 5). The characteristics are parallel to each other, while the strains and stresses are identical everywhere in the domain lying above the characteristics OB, i.e., $p = -p_0 = \text{const}$. Hence, by using (1.4) for $\delta = 0$, (1.17), and (1.19), we have

$$\psi(\xi) = \varphi'(\xi) = -1. \tag{3.6}$$

On the basis of (3.3), (3.4), and (1.16), it is possible to obtain in this same domain

$$v = 2\sqrt{\mu}\,(\mu+1)^{-1}\sqrt{E/\rho}\,(p_0/E)^{(\mu+1)/(2\mu)} = 2\sqrt{\mu}\,(\mu+1)^{-1}u_* = \mathrm{const}.$$

We equate the value of the derivative from (1.3) to this ($\alpha = \beta = 1$)

$$v = \partial u/\partial t = u_*[\varphi(\xi) - \xi\varphi'(\xi)].$$

Taking account of (3.6), we obtain

$$\varphi(\xi) = 2\sqrt{\mu}(\mu+1)^{-1} - \xi. \tag{3.7}$$

The limit of applicability of (3.6) and (3.7) is determined by the location of the characteristic OB. Its equation follows from (3.5) for $t_0 = 0$ and has the form $x/t = a(\varepsilon_0)$. By using (1.5) for $\beta = 1$, the second formulas of (1.2) and (3.4), this expression is reduced to the equality $\xi = \sqrt{\mu}$, governing the domain of action of the formulas mentioned. In the domain lying below the line OB, the characteristics emerge from the origin in the form of divergent lines [1, 4, 5]. The deformation is constant on each characteristic; however, it varies from one characteristic to another. Integrating the first equation in (3.1) for this case, we obtain

$$x/t = a(\varepsilon) = \sqrt{\mu}\sqrt{E/\rho}\,|\varepsilon|^{(\mu-1)/2}.$$

Under further manipulation the left side of this equality is expressed in terms of ξ (1.5) and the right side in terms of $\psi(\xi)$ by means of (1.8) and (1.4). Solving the equality obtained for $\psi(\xi)$, we find

$$\psi(\xi) = -(\sqrt{\mu}/\xi)^{2\mu/(1-\mu)},\quad \varphi' = -(\sqrt{\mu}/\xi)^{2/(1-\mu)}. \tag{3.8}$$

Integrating the second equation, we have

$$\varphi(\xi) = (1-\mu)(1+\mu)^{-1}\mu^{1/(1-\mu)}\xi^{-(1+\mu)/(1-\mu)}. \tag{3.9}$$

The arbitrary constant that appears during integration vanishes, as can be confirmed by using the property of continuity of the functions (3.7) and (3.9) during passage through the point $\xi = \sqrt{\mu}$ determining two domains.

Therefore, for $\mu < 1$, the solution is determined by (3.6) and (3.7) for $\xi \leqq \sqrt{\mu}$, and (3.8) and (3.9) for $\xi \geqq \sqrt{\mu}$. The expressions obtained correspond to (3.2). Infinitely small perturbations are propagated instantaneously over the length of the rod. The explanation for this phenomenon is presented above.

For $\mu > 1$ ($\lambda = 0$) a constant intensity stress wave [3] to which the first formula in (3.2) corresponds is propagated over the rod. According to (1.20), $A_2 = -1$, which yields $\varphi'(\xi) = \psi(\xi) = -1$. We have for the displacement (1.3)

$$u = u_* t\,(A_1 - \xi) = u_* b^{-1}(D_* t - x)\ \text{ for }\ x \leqslant D_* t,\quad u = 0\ \text{ for }\ x > D_* t, \tag{3.10}$$

where $D_* = A_1 b$ is the velocity of wave front propagation. It cannot be determined from the differential equation but is found on the basis of the theorem about the change in momentum. The appropriate formula written for the wave front [4], $p_0 = \rho D_* v$, yields, after substitution of the value $v = \partial u/\partial t$ computed from (3.10), $D_* = \sqrt{E/\rho}(p_0/E)^{(\mu-1)/(2\mu)}$. We take the opportunity to mention a misprint admitted by the author in [13]: In the expression for τ_1 after (4.3), $\sqrt{Dm}$ should be replaced by $\sqrt{D/m}$.

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MODULI OF ELASTICITY OF MATERIALS IN EXTENSION AND COMPRESSION

A. M. Zhukov

UDC 539.3

In recent years there have been extensive developments in the multimodulus theory of elasticity. This theory assumes a material to be homogeneous, but to possess different moduli of elasticity for pure extension and compression in a fixed direction.

Experimental values of the modulus of elasticity for extension E_e and compression E_c were presented in [1]. The reliability of some of these values is doubtful.

For the vitreous plastics KS-30 and AS-30, based on capron and amide resins, respectively, the difference between the moduli at room temperature reaches about 700% of the smaller value. The equipment and methods used for testing were described in [2] (whence this information was taken). Specimens in the form of double spades were used for the extension tests. The equipment used for measurement of deformation was not indicated. Specimens 10 × 10 × 15 mm were tested under compression, with the relative velocity of the reference plates toward each other being 10 mm/min, a value which the authors of [2] erroneously term the deformation rate.

The specimens tested in extension were of a variable cross section. Under tension not only extensive stresses varying over specimen length, but also transverse and tangent stresses were produced. None of this was considered in processing the experimental results.

Due to friction on the faces and pressure of the reference plates, under compression a complex stress state developed within the specimens, inhomogeneous over volume. Calculation of this state would be extremely difficult, but its existence cannot be neglected.

The tests under consideration and the subsequent processing of the data were carried out improperly, producing unreliable data. Incidentally, the curves presented in Fig. 1 of [2] do not have linear initial segments, so that it remains unclear in what manner the moduli of elasticity were determined.

Moduli E_e and E_c for polymethyl methacrylate, taken from [3], were presented in [1]. The differences between the moduli reach 100% of the smaller value.

The stresses acting on the specimen were measured by a photoelectric-optical dynamometer, described in [4], which noted the shortcomings of this device: nonlinearity of the relation between photocurrent and stress, and the necessity of frequent calibration to allow for fatigue

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 4, pp. 128-131, July-August, 1985. Original article submitted June 19, 1984.

 0021-8944/85/2604-0568$09.50